# Fractionally integrated COGARCH processes

Stephan Haug * Claudia Klüppelberg* German Straub*

May 18, 2015

**Abstract**

We construct fractionally integrated continuous time GARCH models, which capture the observed long range dependence of squared volatilities in high-frequency data. We discuss the Molchan-Golosov kernel and the Mandelbrot-van-Ness kernel with respect to their integrability, and the resulting fractional processes with respect to their increments, as they should be stationary and positively correlated. Since this poses problems we resort to moderately long memory problems by choosing a fractional parameter $d \in (-0.5, 0)$ and remove the singularities of the kernel to obtain non-pathological sample path properties. The new fractional COGARCH process has certain positive features like stationarity and algebraically decreasing covariance function.

## 1 Introduction

The goal of this paper is to construct fractionally integrated continuous time GARCH (COGARCH) models in order to capture the long range dependence behaviour of squared volatilities as observed in high-frequency data in finance, but also in turbulence data and others. All stochastic quantities of our paper are defined on a probability space $(\Omega, \mathcal{F}, P)$.

COGARCH models are specified by two equations, the mean and the variance equation. The single source of variation is a Lévy process $L$ with characteristic triplet $(\gamma_L, \sigma^2, \nu_L)$. We refer to [20] for background on Lévy processes. Then the COGARCH(1,1) price process $(G_t)_{t\geq 0}$ is given by

$$dG_t = \sigma_{t-} dL_t, \quad t \geq 0\,,$$

where the squared volatility $\sigma_t^2$ satisfies the SDE

$$d\sigma_t^2 = -\beta_1(\sigma_{t-}^2 - a_0)dt + \alpha_1 \sigma_{t-}^2 d[L,L]_t^{(D)}, \qquad t > 0, \tag{1.1}$$

with parameters $\alpha_0, \alpha_1, \beta_1 > 0$. When $[L,L]_t$ denotes the quadratic variation process of $L$, then

$$[L,L]_t = \sigma^2 t + \sum_{0<s\leq t} \Delta L_s)^2 = \sigma^2 t + [L,L]_t^{(D)}, \quad t \geq 0\,,$$

and $[L,L]_t^{(D)}$ denotes the discrete part of the quadratic variation process of $L$.

The solution of (1.1) is given by

$$\sigma_t^2 = e^{-X_t}\left(\sigma_0^2 + \alpha_0 \beta_1 \int_0^t e^{X_s} ds\right), \qquad t \geq 0, \tag{1.2}$$

*Department of Mathematics, Technische Universität München, 85748 Garching, Boltzmannstrasse 3, Germany, email: {haug , cklu}@tum.de

AMS 2010 subject classification: Primary: 60G10, 60G51, 60H05; Secondary: 91B70

Key words and phrases: fractionally integrated COGARCH, FICOGARCH, long range dependence, fractional subordinator, stationarity, Lévy processes, volatility modelling, stochastic intermittency modelling

with auxiliary process

$$X_t = \beta_1 t - \sum_{s \le t} \log(1 + \alpha_1 (\Delta L_s)^2), \qquad t \ge 0. \tag{1.3}$$

For more background on COGARCH processes cf. [8] and for a recent review [9]. The auxiliary process $X$ is a spectrally negative Lévy process driven by the subordinator $[L, L]^{(D)}$. In order to obtain any long range dependence model for the volatility this driving subordinator has to be modified.

Fractional Lévy processes (fLp) generalize fractional Brownian motion (fBm) in a natural way. It is well-known (e.g. [5] or [19]) that fBm can be defined as a convolution of Brownian motion with a Volterra-type kernel. Two such kernels with fractional parameter $d \in (-0.5, 0.5)$ are the Mandelbrot-van-Ness kernel, which leads to fBm on $\mathbb{R}$ and the Molchan-Golosov kernel, which results in fBm on $\mathbb{R}_+ = [0, \infty)$. Such Gaussian processes have continuous sample paths, stationary increments, and they are self-similar. Moreover, they can model long range dependence for $d > 0$.

For fractional parameter $d \in (-0.5, 0.5)$ the Molchan-Golosov (MG) kernel is defined for $t \ge 0$ as

$$f_d^{MG}(t, s) = c_d (t - s)^d {}_2F_1 \left( -d, d, d + 1, \frac{s - t}{s} \right), \quad s \in [0, t], \tag{1.4}$$

where ${}_2F_1$ is Gauss' hypergeometric function, and the Mandelbrot-van-Ness (MvN) kernel is defined for $t \in \mathbb{R}$ as

$$f_d^{MvN}(t, s) = \tilde{c}_d \left( (t - s)_+^d - (-s)_+^d \right), \qquad s \in \mathbb{R}, \tag{1.5}$$

with constants $c_d = \left( \frac{(2d+1)\Gamma(1-d)}{\Gamma(1+d)\Gamma(1-2d)} \right)^{\frac{1}{2}}$ and $\tilde{c}_d = \frac{1}{\Gamma(d+1)}$, respectively. Replacing the driving Brownian motion by a non-Gaussian Lévy process leads to fractional models with different finite-dimensional distributions (cf. [21]).

Both representations are useful depending on the envisaged application. For instance, the Molchan-Golosov fLp has no infinite history, whereas the Mandelbrot-van-Ness fLp has. More properties have been shown e.g. in [1], [6], [13], and [21].

According to [17] the existence of a fractional subordinator integral with respect to a Lévy process $L$ requires the following properties: $L$ has finite second moment and the kernel $f \in L^1 \cap L^2$. Unfortunately, the MG kernel leads to a fractional subordinator with non-stationary increments and the MvN kernel belongs to $L^1 \cap L^2$ only for negative fractional parameter. With the goal to obtain non-pathological sample paths we define a modified MvN kernel. Using this kernel function we obtain a fractional subordinator, which has a continuous modification and stationary increments. The autocovariance function of these increments decreases with an algebraic rate. This allows us to define a fractionally integrated COGARCH(1,1) volatility process driven by the subordinator $[L, L]^{(D)}$. We also find a stationary version of this new volatility model, which results in a price process with stationary increments.

Our paper is organised as follows. We will discuss fractional subordinator models based on the MG-fLp and the MvN-fLp for $d \in (-0.5, 0.5)$ in Section 2. In Section 3 we present a new modified MvN kernel and investigate its properties. We also define the resulting long memory subordinator and present the sample path properties, its cumulant generating function, and the properties of its increments. Taking as subordinator the quadratic variation process of a finite variance Lévy process allows us to define a fractional COGARCH(1,1) process in Section 4. Our main result in this section is the existence of a stationary version of the variance process, which implies that the price process has stationary increments. We conclude our paper with the straightforward extension of our approach to COGARCH($p, q$) processes.

## 2 Fractional subordinators – naive approaches

The following result will clarify the existence of the fractional subordinator driven integral.

**Proposition 2.1.** *Consider the kernels in* (1.4) *and* (1.5). *Then the following holds.*

*(i) If $d \in (-0.5, 0.5)$ then $f_d^{MG}(t,\cdot) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for all $t > 0$.*

*(ii) Let $t \in \mathbb{R}$. If $d \in (0, 0.5)$, then $f_d^{MvN}(t,\cdot) \in L^2(\mathbb{R})$ but $f_d^{MvN}(t,\cdot) \notin L^1(\mathbb{R})$. If, however, $d \in (-0.5, 0)$ then $f_d^{MvN}(t,\cdot) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.*

*Proof.* For part $(i)$ see [7, Remark 3.3]. For the proof of $(ii)$ see [6, Proposition 2]. □

Consequently, (cf. [17, Theorem 3.3]) we can define fractional subordinators based on the MG kernel for all $d \in (-0.5, 0.5)$ as

$$S_t^d = \int_0^t f_d^{MG}(t,u)\, dS_u, \qquad t \geq 0. \tag{2.1}$$

**Proposition 2.2.** *Let $S = (S_t)_{t\geq 0}$ be a subordinator satisfying $\mathbb{E}(S_1^2) < \infty$. Then $S_t^d$ as defined in* (2.1) *exists for all $t \geq 0$ as limit in probability or in the $L^2(\Omega)$-sense.*

Fractional subordinator models have also been suggested in [2] for a time change process on $[0,\infty)$. Their Example 1 is based on $d > 0$. In this case $f_d^{MG}(\cdot, s)$ is increasing for each $s > 0$ and since they require that $S$ is a strictly increasing subordinator, $S^d$ is a.s. strictly increasing. Hence the integral (2.1) exists as an improper Riemann integral. Further, $S^d$ is adapted to the filtration generated by $S$ such that [16, Section II, Theorem 7] implies that $S^d$ is a semimartingale with characteristics $(S^d, 0, 0)$.

Since we would like to model the long range dependence effect seen in empirical volatilities, we are interested in the case $d > 0$. However, a simple calculation shows that for every $d \in (-0.5, 0.5) \setminus 0$ the expectation $\mathbb{E}(S_t^d) = C_d t^{2d+1}$, which implies immediately that $S^d$ cannot have stationary increments for $d \neq 0$. This makes $S^d$ as driving process of any SDE awkward, since the calculation and estimation of the model parameters would be extremely hard, if at all possible.

Consequently, we turn to the fractional subordinator based on the MvN kernel, which requires a two-sided subordinator. As usual, for any Lévy process $L$, we define a two-sided Lévy process

$$L_t = -L_{-t-}^{(1)} \mathbb{1}_{\{t<0\}} + L_t^{(2)} \mathbb{1}_{\{t\geq 0\}} \tag{2.2}$$

on the whole of $\mathbb{R}$, where $L^{(1)}$ and $L^{(2)}$ are two independent and identically distributed copies of $L$. Then for a two-sided subordinator $S$ we define for $d \in (-0.5, 0)$ (where the integral exists)

$$S_t^d = \int_{\mathbb{R}} f_d^{MvN}(t,u)\, dS_u, \qquad t \in \mathbb{R}. \tag{2.3}$$

However, having the same covariance structure as fBm, $S^d$ has negatively correlated increments for $d \in (-0.5, 0)$. Further, due to the singularities at $s = 0$ and $s = t$ the kernel $f_d^{MvN}(\cdot, s)$ is discontinuous and unbounded for all $s \in \mathbb{R}$, such that $S^d$ has discontinuous and unbounded sample paths with positive probability (cf. [21, Remark 3.3] and [18, Theorem 4]).

# 3 Modified MvN subordinators

There is still a possibility to construct some long-range dependent fractional subordinator with stationary increments following an idea of Brockwell and Marquardt [13], Section 8. In order to define fractional CARMA subordinator models, they suggest a convenient family of moderately long memory fractional Lévy processs generated from the MvN kernel $f_d^{MvN}$ in (1.5). We will adapt their approach in principle. First we consider the MvN-kernel $f_d^{MvN}(t,\cdot)$ for $t \in \mathbb{R}$ and restrict $d$ to be negative, that is

$$f_d^{MvN}(t,s) = \frac{1}{\Gamma(1+d)} \left( (t-s)_+^d - (-s)_+^d \right), \qquad s \in \mathbb{R}. \tag{3.1}$$

From Proposition 2.1 we know that $f_d^{MvN}(t,\cdot) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

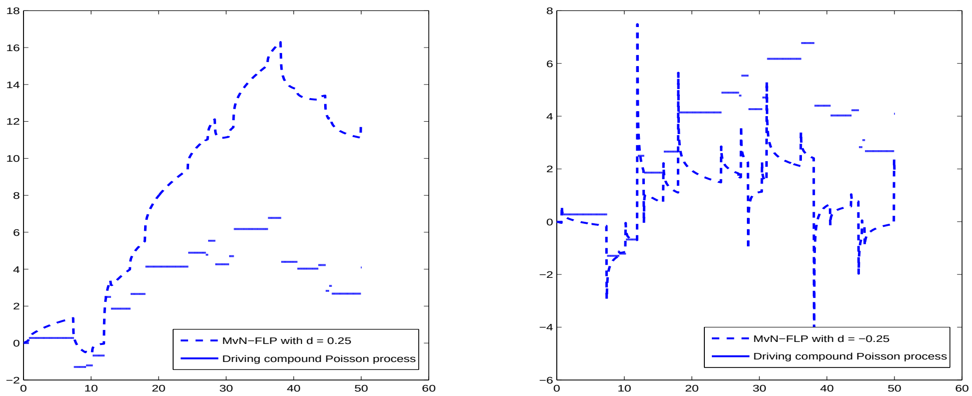


Figure 1: Simulated MvN-fLp (dashed line) with fractional integration parameter $d = 0.25$ (left) and $d = -0.25$ (right) driven by a compound Poisson process (solid line) with rate 0.4 and standard normally distributed jump size.

A fractional Lévy process generated from the kernel (3.1) will still have the same covariance structure as fractional Brownian motion. Hence the increments of such a process will be negatively correlated, see e.g. [11, Section 5.1]. Further, due to the singularity at $s = t$ the function $f_d^{Mvn}(\cdot, s)$ is discontinuous for all $s \in \mathbb{R}$, such that $S^d$ has discontinuous and unbounded sample paths with positive probability (cf. [18, Theorem 4]). To overcome these drawbacks, we will bound the kernel at its singularities.

Observe that the MvN-kernel $f_d^{MvN}(t, \cdot)$ is up to a constant given by $s \mapsto g_d(t-s) - g_d(-s)$ where the function $g_d$ is defined by $g_d(x) := x_+^d$. This suggests to bound $g_d$ at the singularities $s = 0$ and $s = t$ by incorporating a shift $a > 0$, leading to

$$g_{a,d}(x) := (a + x_+)^d, \qquad x \in \mathbb{R}. \tag{3.2}$$

In the following we give the definition of the resulting modification of the Mandelbrot-van-Ness kernel.

**Definition 3.1.** *Let $d < 0$ and $a > 0$. For each $t \in \mathbb{R}$ the* non-normalized modified MvN-kernel *is given by*

$$\begin{aligned} f_{a,d}(t,s) &= g_{a,d}(-s) - g_{a,d}(t-s) \\ &= (a + (-s)_+)^d - (a + (t-s)_+)^d, \qquad s \in \mathbb{R}. \end{aligned} \tag{3.3}$$

**Remark 3.2.** *(a) The normalizing constant of the MG kernel is needed for the integral representation of fBm, and for the MvN kernel the constant is a natural consequence of fractional integration or differentiation. As this is not obvious for our new kernel, we work with the unnormalized version.*

*(b) Observe that besides substituting $g_d$ by $g_{a,d}$, we changed the signs. This way we ensure $f_{a,d}(t, \cdot)$ to be non-negative for $t \geq 0$.*

**Proposition 3.3.** *For $a > 0$ and $d < 0$ consider the modified MvN-kernel $f_{a,d}$ as in* (3.3)*. Then the following holds for all $t \in \mathbb{R}$.*

*(i) $f_{a,d}(t, \cdot)$ is continuous,*

*(ii) $|f_{a,d}(t,s)| \leq a^d$ for all $s \in \mathbb{R}$,*

*(iii) $|f_{a,d}(t,s)| \sim |s|^{d-1}$ as $s \to -\infty$.*

*(iv) $f_{a,d}(t, \cdot) \in L^\delta(\mathbb{R})$ for all $\delta > 1/|d-1|$; in particular, $f_{a,d}(t, \cdot) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.*

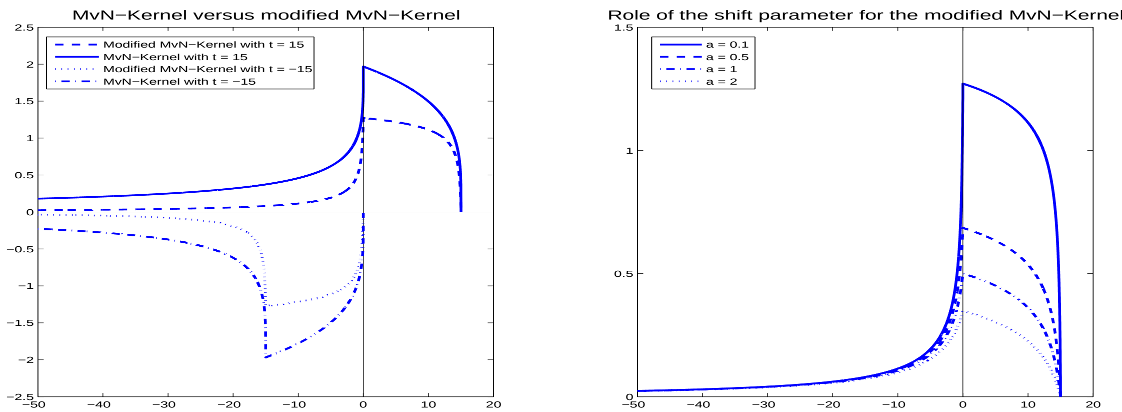


Figure 2: Left: Comparison of the MvN kernel $f_d^{MvN}(t,\cdot)$ and the modified MvN kernel $f_{a,-d}(t,\cdot)$ for $d = 0.25, a = 0.1$. Right: Modified MvN kernel $f_{a,d}(t,\cdot)$ with $d = -0.25$ for different values of the shift parameter $a$ for $t = 15$.

*Proof.* (i) and (ii) are obvious; (iii) is based on the fact that for all $t \in \mathbb{R}$, by a Taylor expansion,

$$\lim_{s\to-\infty} \frac{|f_{a,d}(t,s)|}{|s|^{d-1}} = \lim_{s\to-\infty} \frac{|(a-s)^d - (t+a-s)^d|}{|s|^{d-1}} = \lim_{s\to-\infty} \frac{|dt(a-s)^{d-1}|}{|s|^{d-1}} = |dt|. \tag{3.4}$$

(iv) From (iii) follows that it suffices to check for which $\delta > 0$ the integral $\int_{-\infty}^{N} |s|^{\delta(d-1)} ds < \infty$ for some $N < 0$. This is exactly the case for $\delta > 1/|d-1|$. $\square$

For any two-sided subordinator $S = (S_t)_{t\in\mathbb{R}}$ with $\mathbb{E}(S_1^2) < \infty$, by substituting the MvN-kernel $f_d^{MvN}(t,\cdot)$ in (2.3) with $f_{a,d}(t,\cdot)$ for $d < 0$ and $a > 0$, we define the fractional subordinator $S^{a,d}$ as

$$S_t^{a,d} = \int_{\mathbb{R}} f_{a,d}(t,u)\, dS_u, \qquad t \in \mathbb{R}. \tag{3.5}$$

**Proposition 3.4.** *Let $a > 0$ and $d < 0$. Further denote by $S = (S_t)_{t\in\mathbb{R}}$ a two-sided subordinator satisfying $\mathbb{E}(S_1^2) < \infty$. Then $S_t^{a,d}$ as defined in* (3.5) *exists for all $t \in \mathbb{R}$ as limit in probability and as limit in the $L^2(\Omega)$-sense.*

**Remark 3.5.** *There is some freedom on the choice of an appropriate kernel.*

*(a) For comparison, recall from above that the MvN kernel is for $d \in (-0.5, 0)$ and $t \in \mathbb{R}$ defined as*

$$f_d^{MvN}(t,s) = \frac{1}{\Gamma(1+d)}(g_d(t-s) - g_d(-s)) = \frac{-1}{\Gamma(1+d)} \frac{d}{ds} \int_s^\infty \mathbb{1}_{(0,t)}(v)(v-s)^d dv, \quad t,s \in \mathbb{R},$$

*see [15, Lemma 1.1.3]. This kernel can generate fBm and fLm on $\mathbb{R}$ for symmetric driving processes.*

*(b) The approach in [4] to construct a FICARMA process driven by a subordinator would suggest to use the kernel*

$$\frac{-1}{\Gamma(1+d)} \frac{d}{ds} \int_s^\infty \mathbb{1}_{(0,t)}(v) \min(a^d, (v-s)^d) dv, \quad t,s \in \mathbb{R},$$

*for $d \in (-0.5, 0)$ and some $a > 0$. For us it was computational advantageous to bound $g_d$ and not the integrand in the above representation.*

*(c) [14] suggest a tempered fractional kernel of the form*

$$e^{-\lambda(t-s)_+}(t-s)_+^d - e^{-\lambda(-s)_+}(-s)_+^d dv, \quad t,s \in \mathbb{R},$$

*for $d \in (-0.5, 0.5)$ and $\lambda > 0$, which also has an integral representation using tempered fractional integrals, see [14, Definition 2.1] for details. It belongs to $L^p$ for all $p \geq 1$. The autocovariance function of the increments of the corresponding tempered fractional process decreases for small lags algebraically, but for large lags exponentially. It is therefore called a semi-long range dependence model.*

(d) *[10] generalize fractional Lévy processes by allowing the kernel to be regularly varying, which allows to prove functional central limit theorems for scaled Ornstein-Uhlenbeck processes driven by such generalized fractional Lévy processes.*

## 3.1 Sample path properties

The fractional subordinator $S^{a,d}$ has smooth sample paths in the following sense.

**Proposition 3.6.** *Let $d \in (-0.5, 0)$ and $a > 0$. Further let $S = (S_t)_{t \in \mathbb{R}}$ be a two-sided subordinator satisfying $\mathbb{E}(S_1^2) < \infty$. Then $S^{a,d}$ as defined in* (3.5) *has the following properties.*

(i) *$(S_t^{a,d})_{t \geq 0}$ has a.s. continuous sample paths of finite variation.*

(ii) *$(S_t^{a,d})_{t \geq 0}$ constitutes a semimartingale (with respect to the filtration generated by $S$) with characteristics given by $(S^{a,d}, 0, 0)$.*

(iii) *If $S$ has strictly increasing sample paths, i.e. characteristic triple $(\gamma_S, 0, \nu_S)$ such that either $\gamma > 0$ or $\nu(0, \infty) = \infty$, then $(S_t^{a,d})_{t \geq 0}$ has strictly increasing sample path.*

*Proof.* (i) Recall the representation of the two-sided subordinator $S_t = -S^{(1)}_{-t-} \mathbb{1}_{\{t<0\}} + S^{(2)}_t \mathbb{1}_{\{t \geq 0\}}$ for independent identically distributed subordinators $S^{(1)}$ and $S^{(2)}$ with $S_0 = 0$. Define $\mathcal{F}_t := \sigma\{S_t : t \geq 0\}$, hence, $\mathcal{F}_0$ is the sigma algebra generated by $S^{(1)}$. Then $(S_t^{a,d})_{t \geq 0}$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$. Now we can write

$$S_t^{a,d} = S_0^{a,d} + \int_0^t (a^d - (a + t - u)^d) dS_u, \quad t \geq 0.$$

Since the kernel function $f_{a,d}(t, \cdot)$ and almost all paths of $S$ have finite variation, we can apply partial integration to such paths, which yields

$$S_t^{a,d} = S_0^{a,d} + d \int_0^t (a + t - u)^{d-1} S_u du, \quad t \geq 0.$$

This implies that $(S_t^{a,d})_{t \geq 0}$ has a.s. continuous and finite variational sample paths.
(ii) Together with (i) [16, Section II, Theorem 7] implies that $(S_t^{a,d})_{t \geq 0}$ is a semimartingale with the given characteristics.
(iii) This is a consequence of Prop. 1 of [2]. □

## 3.2 Cumulant generating function and moments

Let $S$ be a subordinator without drift, then its cumulant generating function is given by $\ln \mathbb{E}(e^{\theta S_1}) = \psi(\theta) = \int_0^\infty (1 - e^{\theta z}) \nu_S(dz)$, where $\nu_S$ is the Lévy measure of $S$. Proposition 2.6 of [17] gives the cumulant generating function of $S^{a,d}$ as

$$\ln \mathbb{E}(e^{\theta S_t^{a,d}}) = \ln \mathbb{E}\Big( \exp\Big\{ \theta \int_{\mathbb{R}} f_{a,d}(t,u)\, dS_u \Big\} \Big) = \int_{\mathbb{R}} \int_0^\infty (1 - e^{\theta f_{a,d}(t,u) z}) \nu_S(dz) du.$$

The $k$-th cumulant of $S_t^{a,d}$ is given by

$$\kappa^k(S_t^{a,d}) = \frac{d^k}{d\theta^k} \ln \mathbb{E}(e^{\nu S_t^{a,d}}) \Big|_{\nu=0}.$$

For the $k$-th derivative in 0 we obtain, provided the corresponding Lévy moment exists

$$\kappa^k(S_t^{a,d}) = \int_0^\infty z^k \nu_S(dz) \int_{\mathbb{R}} f_{a,d}^k(t,u)du = \kappa^k(S_1) \int_{\mathbb{R}} f_{a,d}^k(t,u)du.$$

From this we calculate the mean and variance as

$$\mathbb{E}(S_t^{a,d}) = \mathbb{E}(S_1) \int_{\mathbb{R}} f_{a,d}(t,u)du,$$
$$\mathbb{V}\mathrm{ar}(S_t^{a,d}) = \mathbb{V}\mathrm{ar}(S_1) \int_{\mathbb{R}} f_{a,d}^2(t,u)du.$$

## 3.3 Properties of increments

Next we ask whether $f_{a,d}$ serves its purpose in the sense that the increments of $S^{a,d}$ exhibit an *appropriate* dependence structure. We start with an auxiliary result.

**Lemma 3.7.** *Let $a > 0$ and $d \in (-0.5, 0)$. Then the modified MvN-kernel $f_{a,d}$ as in* (3.3) *satisfies for $t > 0$*

$$\int_{\mathbb{R}} f_{a,d}^2(t,u)\, du = C + a^{2d}\, t - \frac{2a^d}{d+1}(t+a)^{d+1} + \frac{1}{2d+1}(t+a)^{2d+1} + c(t)t^{2d+1}$$

*with*

$$C = a^{2d+1}\Big(\frac{2}{d+1} - \frac{1}{2d+1}\Big) \quad \text{and} \quad c(t) = \int_{-\infty}^{-a/t} \Big[(1-y)^d - (-y)^d\Big]^2\, dy, \tag{3.6}$$

*and*

$$\lim_{t\to\infty} c(t) = \frac{\Gamma(d+1)}{\Gamma(2d+2)\sin(\pi(d+0.5))} + \frac{1}{2d+1}. \tag{3.7}$$

*Proof.* By substituting $y := (u-a)/t$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}} f_{a,d}^2(t,u)\, du &= \int_{-\infty}^0 \Big[(a-u)^d - (a+t-u)^d\Big]^2\, du + \int_0^t \Big[a^d - (a+t-u)^d\Big]^2\, du \\
&= \int_{-\infty}^0 t^{2d}\left[\Big(1-\frac{u-a}{t}\Big)^d - \Big(-\frac{u-a}{t}\Big)^d\right]^2 du + \int_0^t \Big[a^d - (a+t-u)^d\Big]^2\, du \\
&= t^{2d+1}\int_{-\infty}^{-a/t} \Big[(1-y)^d - (-y)^d\Big]^2\, dy + \int_0^t a^{2d} - 2a^d\,[t+a-u]^d + [t+a-u]^{2d}\, du \\
&= t^{2d+1}c(t) + a^{2d}\, t + \frac{2a^d}{d+1}\,[t+a-u]^{d+1}\Big|_0^t - \frac{1}{2d+1}\,[t+a-u]^{2d+1}\Big|_0^t.
\end{aligned}$$

Further note that for the normalization constant $1/\Gamma(d+1)$ in (3.1) we obtain

$$\int_{-\infty}^1 \Big[(1-y)_+^d - (-y)_+^d\Big]^2\, dy = \Gamma(d+1)\int_{-\infty}^1 \Big(f_d^{MvN}(1,y)\Big)^2\, dy = \frac{\Gamma(d+1)}{\Gamma(2d+2)\sin(\pi(d+0.5))}.$$

Consequently,

$$\begin{aligned}
c(t) &= \frac{\Gamma(d+1)}{\Gamma(2d+2)\sin(\pi(d+0.5))} - \int_{-a/t}^0 \Big[(1-y)^d - (-y)^d\Big]^2\, dy - \int_0^1 (1-y)^{2d}\, dy \\
&= \frac{\Gamma(d+1)}{\Gamma(2d+2)\sin(\pi(d+0.5))} - \int_{-a/t}^0 \Big[(1-y)^d - (-y)^d\Big]^2\, dy + \frac{1}{2d+1}.
\end{aligned}$$

Since $\int_{-a/t}^0 \Big[(1-y)^d - (-y)^d\Big]^2\, dy \to 0$ as $t \to \infty$, the assertion holds. $\square$

**Proposition 3.8.** *Let $a > 0$ and $d \in (-0.5, 0)$. Further denote by $S = (S_t)_{t\in\mathbb{R}}$ a two-sided subordinator satisfying $\mathbb{E}(S_1^2) < \infty$. Let $S^{a,d}$ be defined in* (3.5)*, then the increments of $S^{a,d}$ have the following properties.*

*(i) $S^{a,d}$ has stationary increments.*

*(ii) Let $r > 0$ be fixed and $s + r \leq t$ such that $t - s = hr$ for some $h > 0$. Then the two increments $S^{a,d}_{t+r} - S^{a,d}_t$ and $S^{a,d}_{s+r} - S^{a,d}_s$ of length $r$ have covariance*

$$\gamma_r(h) := \mathbb{C}\mathrm{ov}\left(S^{a,d}_{s+(h+1)r} - S^{a,d}_{s+hr}, S^{a,d}_{s+r} - S^{a,d}_s\right),$$

*which satisfies*

$$\gamma_r(h) \sim \mathbb{V}\mathrm{ar}\big(S^{a,d}_1\big)|d|a^d r^2\ (hr+a)^{d-1}, \quad h \to \infty. \tag{3.8}$$

*Proof.* (i) For $n \in \mathbb{N}$, $t_0 < t_1 < \cdots < t_n$ and $a_1, \ldots, a_n \in \mathbb{R}$ we use the Cramér-Wold device and calculate

$$\begin{aligned}\sum_{i=1}^n a_i(S^{a,d}_{t_i+h} - S^{a,d}_{t_{i-1}+h}) &= \sum_{i=1}^n a_i \int_{\mathbb{R}} (g_{a,d}(t_{i-1}+h-u) - g_{a,d}(t_i+h-u))dS_u \\ &\stackrel{d}{=} \sum_{i=1}^n a_i \int_{\mathbb{R}} (g_{a,d}(t_{i-1}-v) - g_{a,d}(t_i-v))dS_v \\ &= \sum_{i=1}^n a_i(S^{a,d}_{t_i} - S^{a,d}_{t_{i-1}}),\end{aligned}$$

where we have used the stationary increments of $S$.
(ii) We introduce the notation $\tilde{S}_t := S_t - \mathbb{E}(S_t)$ and $\tilde{S}^{a,d}_t := \int_{\mathbb{R}} f_{a,d}(t,u)\, d\tilde{S}_u$. For $t, s \geq 0$ we calculate

$$\begin{aligned}\mathbb{C}\mathrm{ov}(S^{a,d}_t, S^{a,d}_s) &= \mathbb{C}\mathrm{ov}(\tilde{S}^{a,d}_t, \tilde{S}^{a,d}_s) = \mathbb{E}\big(\tilde{S}^{a,d}_t \tilde{S}^{a,d}_s\big) = \frac{1}{2}\left(\mathbb{E}\big((\tilde{S}^{a,d}_t)^2\big) + \mathbb{E}\big((\tilde{S}^{a,d}_s)^2\big) - \mathbb{E}\big((\tilde{S}^{a,d}_t - \tilde{S}^{a,d}_s)^2\big)\right) \\ &= \frac{1}{2}\left(\mathbb{E}\big((\tilde{S}^{a,d}_t)^2\big) + \mathbb{E}\big((\tilde{S}^{a,d}_s)^2\big) - \mathbb{E}\big((\tilde{S}^{a,d}_{t-s})^2\big)\right).\end{aligned}$$

In the last step we have used that the increments are stationary. Furthermore,

$$\mathbb{E}\big((\tilde{S}^{a,d}_t)^2\big) = \mathbb{V}\mathrm{ar}\big(\tilde{S}^{a,d}_t\big) = \mathbb{V}\mathrm{ar}\,(S_1) \int_{\mathbb{R}} f^2_{a,d}(t,u)\, du,$$

such that by the linearity of the covariance operator,

$$\begin{aligned}\gamma_r(h) &= \mathbb{C}\mathrm{ov}(S^{a,d}_{s+(h+1)r}, S^{a,d}_{s+r}) - \mathbb{C}\mathrm{ov}(S^{a,d}_{s+(h+1)r}, S^{a,d}_s) - \mathbb{C}\mathrm{ov}(S^{a,d}_{s+hr}, S^{a,d}_{s+r}) + \mathbb{C}\mathrm{ov}(S^{a,d}_{s+hr}, S^{a,d}_s) \\ &= \frac{1}{2}\left(\mathbb{E}\big((\tilde{S}^{a,d}_{(h+1)r})^2\big] + \mathbb{E}\big((\tilde{S}^{a,d}_{(h-1)r})^2\big] - 2\mathbb{E}\big((\tilde{S}^{a,d}_{hr})^2\big]\right) \\ &= \frac{1}{2}\mathbb{V}\mathrm{ar}\,(S_1)\left[\int_{\mathbb{R}} f^2_{a,d}\big((h+1)r, u\big)\, du + \int_{\mathbb{R}} f^2_{a,d}\big((h-1)r, u\big)\, du - 2\int_{\mathbb{R}} f^2_{a,d}(hr, u)\, du\right].\end{aligned}$$

Now, using Lemma 3.7 we obtain

$$\begin{aligned}\gamma_r(h) = \frac{1}{2}\mathbb{V}\mathrm{ar}\,(S_1)\Big[&-\frac{2a^d}{d+1}\Big(((hr+a)+r)^{d+1} + ((hr+a)-r)^{d+1} - 2(hr+a)^{d+1}\Big) \\ &+ \frac{1}{2d+1}\Big(((hr+a)+r)^{2d+1} + ((hr+a)-r)^{2d+1} - 2(hr+a)^{2d+1}\Big) \\ &+ c(hr+r)(hr+r)^{2d+1} + c(hr-r)(hr-r)^{2d+1} - 2c(hr)(hr)^{2d+1}\Big],\end{aligned}$$

where $c(t)$ is defined as in (3.6) and according to (3.7) converges for $t \to \infty$ to a positive constant, which we denote by $c$. Consequently, a Taylor expansion and the fact that $\lim_{h\to\infty} c(hr \pm r)/c(hr) \to 1$ gives for $h \to \infty$

$$\begin{aligned}
\gamma_r(h) = \frac{1}{2}\mathbb{V}\mathrm{ar}\,(S_1)\Big[ &-\frac{2a^d}{d+1}(hr+a)^{d+1}\left((d+1)d\frac{r^2}{(hr+a)^2} + \mathcal{O}\left(\frac{1}{(hr+a)^4}\right)\right) \\
&+ \frac{(hr+a)^{2d+1}}{2d+1}\left((2d+1)2d\frac{r^2}{(hr+a)^2} + \mathcal{O}\left(\frac{1}{(hr+a)^4}\right)\right) \\
&+ c\left((2d+1)2d\frac{r^2}{(hr)^2} + \mathcal{O}\left(\frac{1}{(hr)^4}\right)\right)\Big] \\
\sim \mathbb{V}\mathrm{ar}\,(S_1)\,(-d)a^d r^2\,(hr+a)^{d-1}.
\end{aligned}$$

□

**Remark 3.9.** *(a) The result* (3.8) *implies that asymptotically the increments of $S^{a,d}$ are positively correlated.*

*(b) The increments of $S^{a,d}$ do not have long memory in the standard sense, since the covariance function is integrable. However, it decreases algebraically, and for $d$ close to zero we approximately obtain long memory. This is in analogy to the asymptotic rate of decay of the modified CARMA kernel $g_{a,d}$ in the case of a subordinator-driven CARMA processes in Section 8 of [4].*

# 4 The fractionally integrated COGARCH process

Taking the subordinator $S = [L,L]^{(D)}$ and its fractional subordinator $S^{a,d}$ with the modified MvN kernel (3.1) as driving process in (1.3), we can now define the *fractionally integrated* COGARCH$(1,1)$ process.

**Definition 4.1** (FICOGARCH$(1,d,1)$)**.** *Let $\alpha_0, \alpha_1, \beta_1 > 0$ and $d \in (-0.5, 0)$. Assume $L$ to be a Lévy process with $\mathbb{E}(L_1^4) < \infty$. Define the price process $G$ with some initial value $G_0$ and*

$$dG_t = \sigma_{t-} dL_t \qquad t > 0, \tag{4.1}$$

*where the squared volatility $(\sigma_t^2)_{t\geq 0}$ is given as the solution of the SDE*

$$d\sigma_t^2 = -\beta_1(\sigma_t^2 - \alpha_0)\,dt + \alpha_1\sigma_t^2\,dS_t^{a,d}. \tag{4.2}$$

*The process $S^{a,d}$ is the fractional subordinator defined in* (3.5) *with modified MvN kernel* (3.3) *and subordinator $S := [L,L]^{(D)}$, where $[L,L]^{(D)}$ is the discrete part of the quadratic variation of $L$. The model* (4.1) *with* (4.2) *is called* fractionally integrated COGARCH$(1,1)$ process with fractional integration parameter *$d$ or* FICOGARCH$(1,d,1)$. *The stochastic volatility model* (4.2) *is called* FICOGARCH$(1,d,1)$ volatility process with fractional integration parameter *$d$.*

We can state the solution of the SDE (4.2) explicitly.

**Proposition 4.2.** *Consider the FICOGARCH$(1,d,1)$ volatility process as in* (4.2)*. Then for almost all sample paths the pathwise solution of the SDE* (4.2) *with initial value $\sigma_0^2 > 0$ is given by*

$$\sigma_t^2 = e^{-X_t}\left(\sigma_0^2 + \alpha_0\beta_1\int_0^t e^{X_s}\,ds\right), \qquad t \geq 0, \tag{4.3}$$

*with*

$$X_t = \beta_1 t - \alpha_1 S_t^{a,d}, \qquad t \geq 0.$$

*Proof.* Since by Proposition 3.6 (1) $(S^{a,d})_{t\geq 0}$ has a.s. continuous and finite variational sample paths, also $(X)_{t\geq 0}$ has and we can apply integration by parts to obtain

$$\begin{aligned} d\sigma_t^2 &= \Big(\sigma_0^2 + \alpha_0\beta_1 \int_0^t e^{X_s} ds\Big)\, d\big(e^{-X_t}\big) + e^{-X_t} d\left(\sigma_0^2 + \alpha_0\beta_1 \int_0^t e^{X_s} ds\right) \\ &= \sigma_t^2 d\big(-\beta_1 t + \alpha_1 S_t^{a,d}\big) + e^{-X_t} e^{X_t} \alpha_0 \beta_1 dt \\ &= -\beta_1(\sigma_t^2 - \alpha_0)dt + \alpha_1 \sigma_t^2 dS_t^{a,d}. \end{aligned}$$

Thus, (4.3) satisfies the SDE (4.2). □

In Figure 3 we compare the sample paths of two FICOGARCH$(1, d, 1)$ processes with respect to different choices of $d \in (-0.5, 0)$. The left column shows the price, return and volatility process of a FICOGARCH$(1, -0.01, 1)$ process along with the sample autocorrelation function (acf) of the volatility process. The right column depicts the same quantities of a FICOGARCH$(1, -0.40, 1)$ process, which is driven by the same Lévy process. From Proposition 3.8 we would expect a slower decay of the acf of the volatility process for larger values of $d$. This is confirmed by the two sample acfs in the bottom row of Figure 3. For simulating the fractional Lévy process we approximated the process by the corresponding Riemann sums as explained in [12, Section 2.4].

## 4.1 Stationarity of the FICOGARCH(1,d,1)

To formulate the stationary version of the FICOGARCH$(1, d, 1)$ model, we have to extend it to the whole of $\mathbb{R}$. Consequently, in Definition 4.1 we choose a two-sided Lévy process $(L_t)_{t\in\mathbb{R}}$ as in (2.2). Then the stochastic differential equation (4.2) has solution

$$\begin{aligned} \sigma_t^2 &= e^{-X_t}\Big(\sigma_0^2 + \alpha_0\beta_1 \int_0^t e^{X_s} ds\Big), \quad t \in \mathbb{R}, \\ X_t &= \beta_1 t - \alpha_1 S_t^{a,d}, \quad t \in \mathbb{R}. \end{aligned} \tag{4.4}$$

Since the subordinator $S = [L, L]^{(D)}$ has a modification with finite variational sample paths, also the fractional subordinator $S^{a,d}$ has one, and hence $X$. Moreover, by Proposition 3.8 (i), $S^{a,d}$ has stationary increments. Note that

$$S_t^{a,d} = \begin{cases} \int_{-\infty}^t \Big((a+(-s))^d - (a+(t-s))^d\Big) dS_s + \int_t^0 \Big((a+(-s))^d - a^d\Big) dS_s & \text{for } t < 0, \\ \int_{-\infty}^0 \Big((a+(-s))^d - (a+(t-s))^d\Big) dS_s + \int_0^t \Big(a^d - (a+(t-s))^d\Big) dS_s & \text{for } t \geq 0. \end{cases}$$

The following is an analog of Theorem 3.7 of [13] for a fractional Lévy process.

**Proposition 4.3.** *Let $L$ be a two-sided Lévy process with $\mathbb{E}(L_1^2) < \infty$. Let $d \in (-0.5, 0)$ and $a > 0$. Then $S^{a,d}$, defined in* (3.5) *with subordinator $S = [L, L]^{(D)}$, has a modification that equals the improper Riemann integral*

$$S_t^{a,d} = d \int_{\mathbb{R}} \Big((a + (-s)_+)^{d-1} - (a + (t-s)_+)^{d-1}\Big) S_s ds - d a^{d-1} \int_0^t S_s ds, \quad t \in \mathbb{R}. \tag{4.5}$$

*Moreover, $S^{a,d}$ is continuous in $t$.*

*Proof.* First note that for every finite variational sample path of $S^{a,d}$ we can apply partial integration and obtain for $t > 0$

$$\begin{aligned} S_t^{a,d} &= \lim_{u\to-\infty} \int_u^0 \Big((a+(-s))^d - (a+(t-s))^d\Big) dS_s + \int_0^t \Big(a^d - (a+(t-s))^d\Big) dS_s \\ &= -(a^d - (a+t)^d) S_0 - d \int_0^t (a+(t-s))^{d-1} S_s ds + (a^d - (a+t)^d) S_0 \end{aligned}$$

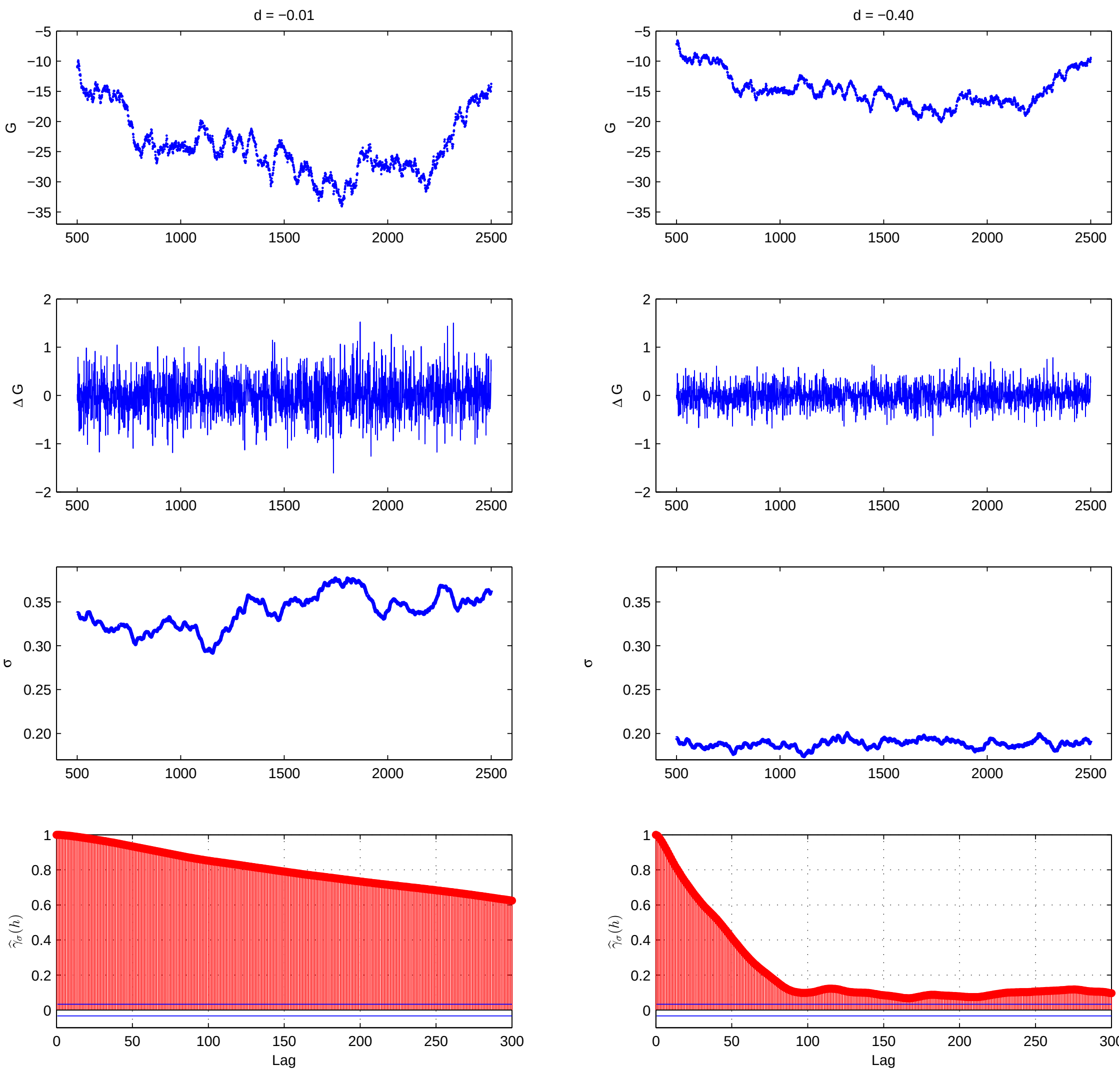


Figure 3: Simulation of the modified MvN-FICOGARCH price process $G$ (top) with corresponding return process $\Delta G$ (second row), volatility process $\sigma$ (third row) and sample acf $\widehat{\gamma}_\sigma$ of $\sigma$ (bottom) driven by a compound Poisson process with rate 5 and normally distributed jump sizes with mean zero and variance one half. The model parameters are $\alpha_0 = 0.0195, \alpha_1 = 0.0105, \beta_1 = 0.0513$ and $a = 1$. The fractional parameter $d$ is equal to $-0.01$ (left) and $-0.4$ (right).

$$
\begin{aligned}
&- \lim_{u \to -\infty} \Big( (a+(-u))^d - (a+(t-u))^d \Big) S_u \\
&+ \lim_{u \to -\infty} d \int_u^0 \Big( (a+(-s))^{d-1} - (a+(t-s))^{d-1} \Big) S_s ds
\end{aligned} \tag{4.6}
$$

Now recall that by the SLLN $\frac{S_s}{|s|} \overset{\text{a.s.}}{\to} \mathbb{E}(L_1^2)$ as $s \to -\infty$. Hence, the integrand in (4.6) satisfies for $s \to -\infty$

$$
\Big( (a+(-s))^{d-1} - (a+(t-s))^{d-1} \Big) |S_s| \sim \mathbb{E}(L_1^2)(d-1)t(-s)^{d-1} \to 0.
$$

Thus the integral (4.6) converges finitely for $u \to -\infty$. This yields

$$
S_t^{a,d} = d \int_{-\infty}^0 \Big( (a-s)^{d-1} - (a+(t-s))^{d-1} \Big) S_s ds - d \int_0^t (a+(t-s))^{d-1} S_s ds.
$$

For $t \le 0$ we get

$$
S_t^{a,d} = d \int_{-\infty}^t \Big( (a-s)^{d-1} - (a+(t-s))^{d-1} \Big) S_s ds + d \int_t^0 (a-s)^{d-1} S_s ds,
$$

Continuity follows from dominated convergence. $\square$

**Proposition 4.4.** *Let $S^{a,d}$ be as above with $d \in (-0.5, 0)$. Then* $\lim_{v \to -\infty} \frac{|S_t^{a,d} - S_v^{a,d}|}{|t-v|} = 0$ *a.s.*

*Proof.* Wlog assume that $v < t < 0$ for fixed $t$. Then

$$
\begin{aligned}
|S_t^{a,d} - S_v^{a,d}| \le& |d| \Bigg( \int_{-\infty}^v \Big| (a+(v-s))^{d-1} - (a+(t-s))^{d-1} \Big| S_s ds \\
&+ \int_v^t \Big| (a+(-s))^{d-1} - (a+(t-s))^{d-1} \Big| S_s ds \\
&+ \int_v^t \Big| (a+(-s))^{d-1} \Big| [L,L]_s ds \Bigg) =: |d| \left( J_1 + J_2 + J_3 \right).
\end{aligned}
$$

To estimate $J_1$ we set $u := t - s$ and use a first order Taylor expansion to obtain for $|\theta| < 1$

$$
\begin{aligned}
\frac{J_1}{|t-v|} &\le \frac{M}{|t-v|} \int_{t-v}^\infty \Big| (a+u+(v-t))^{d-1} - (a+u)^{d-1} \Big| (u+|t|) du \\
&\le \frac{M}{|t-v|} \int_{t-v}^\infty \Big| (d-1)(v-t)(a+u+\theta(v-t))^{d-2} \Big| (u+|t|) du \\
&= M|d-1|\theta \left( |t| \int_{t-v}^\infty (a+u+\theta(v-t))^{d-2} du + \int_{t-v}^\infty (a+u+\theta(v-t))^{d-2} u du \right) \\
&\le M\theta \Big( |t|(a-\theta(t-v)+t-v)^{d-1} + (a-\theta(t-v)+t-v)^{d-2} \frac{1}{2}(t-v)^2 \Big) \\
&= M\theta \Big( |t|(a+(1-\theta)(t-v))^{d-1} + (a+(1-\theta)(t-v))^{d-2} \frac{1}{2}(t-v)^2 \Big) \\
&\to \quad 0, \quad v \to -\infty.
\end{aligned}
$$

To estimate $J_2$ we set again $u := t - s$ and use a first order Taylor expansion to obtain for $|\theta| < 1$

$$
\begin{aligned}
\frac{J_2}{|t-v|} &\le \frac{M}{|t-v|} \int_{t-v}^\infty \Big| (a+u-t)^{d-1} - (a+u)^{d-1} \Big| (u+|t|) du \\
&\le \frac{M|d-1||t|}{|t-v|} \int_{t-v}^\infty (a+u+\theta|t|)^{d-2} (u+|t|) du
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{M|d-1||t|}{|t-v|}\left(\int_{t-v}^{\infty}(a+u+\theta|t|)^{d-2}udu + |t|\int_{t-v}^{\infty}(a+u+\theta|t|)^{d-2}du\right)\\
&\le \frac{M|t|}{|t-v|}\left(\frac{|d-1|}{|d|}(t-v)^d + |d-1||t|(a+t-v+\theta|t|)^{d-1}\right)\\
&\to \quad 0, \quad v \to -\infty.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
J_3 &\le \int_v^t \left|(a+(-s))^{d-1}s\right|ds = \int_0^{|v|-|t|}(a+u)^{d-1}(u+|t|)du\\
&\le \int_0^{|v|-|t|}u^d du + |t|\int_0^{|v|-|t|}(a+u)^{d-1} = \frac{1}{d+1}(|v|-|t|)^{d+1} + \frac{|t|}{d}(a+|v|-|t|)^d - a^d).
\end{aligned}
$$

Hence,

$$
\frac{J_3}{|v|-|t|} \le \frac{1}{d+1}(|v|-|t|)^d + \frac{|t|}{d}\frac{(a+|v|-|t|)^d - a^d}{|v|-|t|}.
$$

For $v \to -\infty$, the first term tends to 0 since $d<0$. By (3.4) we have $(a+|v|-|t|)^d - a^d \sim |t|d(|v|-|t|)^{d-1}$, which implies that $J_3$ also tends to 0 as $v \to -\infty$. □

**Lemma 4.5.** *The integral $\int_v^t e^{-(X_t - X_s)}ds$ exists a.s. for $v \to -\infty$.*

*Proof.* Note that for $C \in (0, \beta_1/\alpha_1)$ we estimate, using Proposition 4.4,

$$
\begin{aligned}
\int_v^t e^{-(X_t-X_s)}ds &\le \int_v^t \exp\Big\{-\beta_1(t-s) - \alpha_1\frac{S_t^{a,d}-S_s^{a,d}}{|t-s|}|t-s|\Big\}ds\\
&= \int_v^t \exp\Big\{-(t-s)\Big(\beta_1 + \alpha_1\frac{S_t^{a,d}-S_s^{a,d}}{|t-s|}\Big)\Big\}ds \qquad \le \int_v^t \exp\Big\{-(t-s)\Big(\beta_1 - \alpha_1 C\Big)\Big\}ds,
\end{aligned}
$$

and the last integral exists for $v \to -\infty$. □

Next we prove that the process $(\widetilde{\sigma}_t^2)_{t\in\mathbb{R}} := \left(\alpha_0\beta_1\int_{-\infty}^t e^{-(X_t-X_s)}ds\right)_{t\in\mathbb{R}}$ is stationary.

**Lemma 4.6.** *For all $t_1 < \cdots < t_m$, $m \in \mathbb{N}$, $h \in \mathbb{R}$,*

$$
(\widetilde{\sigma}_{t_1}^2, \ldots, \widetilde{\sigma}_{t_m}^2) \stackrel{d}{=} (\widetilde{\sigma}_{t_1+h}^2, \ldots, \widetilde{\sigma}_{t_m+h}^2).
$$

*Proof.* We use the Cramér-Wold device and calculate for $a_1, \ldots, a_m \in \mathbb{R}$ and $m \in \mathbb{N}$,

$$
\begin{aligned}
\frac{1}{\alpha_0\beta_1}\sum_{i=1}^m a_i\widetilde{\sigma}_{t_i+h}^2 &= \sum_{i=1}^m a_i\int_{-\infty}^{t_i+h}\exp\Big\{-\beta_1(t_i+h-s) + \alpha_1(S_{t_i+h}^{a,d} - S_s^{a,d})\Big\}ds\\
&= \sum_{i=1}^m a_i\int_{-\infty}^{t_i}\exp\Big\{-\beta_1(t_i-v) + \alpha_1(S_{t_i+h}^{a,d} - S_{v+h}^{a,d})\Big\}dv\\
&\stackrel{d}{=} \sum_{i=1}^m a_i\int_{-\infty}^{t_i}\exp\Big\{-\beta_1(t_i-v) + \alpha_1(S_{t_i}^{a,d} - S_v^{a,d})\Big\}dv\\
&= \frac{1}{\alpha_0\beta_1}\sum_{i=1}^m a_i\widetilde{\sigma}_{t_i}^2,
\end{aligned}
$$

by the stationary increments of $S^{a,d}$. □

The main theorem follows now from the preceding results.

**Theorem 4.7.** *Let the squared FICOGARCH$(1,d,1)$ volatility process $(\sigma_t^2)_{t\in\mathbb{R}}$ be given as in* (4.4) *with $\sigma_0^2 = \alpha_0\beta_1\int_{-\infty}^0 e^{X_s}ds$ independent of $(L_t)_{t\geq 0}$, which satisfies $\mathbb{E}(L_1^4) < \infty$, and parameters as in Definition 4.1. Then $(\sigma_t^2)_{t\geq 0}$ is strictly stationary. Moreover, the price process $(G_t)_{t\geq 0}$ as defined in* (4.1) *has stationary increments.*

Our new approach applies immediately also to higher order COGARCH models.

**Remark 4.8.** *The COGARCH$(p,q)$ process was introduced in [3]. By using the same approach we can generalise the FICOGARCH$(1,d,1)$ model in a straightforward manner to its higher order analog.*

*Let $p$ and $q$ be integers such that $q \geq p \geq 1$. Further let $\alpha_0, \alpha_1, \ldots, \alpha_p \in \mathbb{R}, \beta_1, \ldots, \beta_q \in \mathbb{R}, \alpha_p \neq 0, \beta_q \neq 0$ and $\alpha_{p+1} = \cdots = \alpha_q = 0$. Then we define the $q\times q$ matrix $\mathcal{B}$ and the vectors $\mathbf{a}$ and $\mathbf{1}_q$ by*

$$\mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & 1 \\ -\beta_1 & -\beta_{q-1} & -\beta_{q-2} & \cdots & -\beta_1 \end{pmatrix}, \ \mathbf{a} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{q-1} \\ \alpha_q \end{pmatrix}, \ \mathbf{1}_q = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

*with $\mathcal{B} := -\beta_1$ if $q = 1$. Then for a two-sided Lévy proceess $(L_t)_{t\in\mathbb{R}}$ as in* (2.2) *with $\mathbb{E}(L_1^4) < \infty$, we define the squared volatility process $(\sigma_t^2)_{t\geq 0}$ with parameters $\mathcal{B}, \mathbf{a}, \alpha_0$ by*

$$\sigma_t^2 = \alpha_0 + \mathbf{a}^\top \mathbf{Y}_t, \qquad t \geq 0,$$

*where the state process $(\mathbf{Y}_t)_{t\geq 0}$ is the unique solution of the SDE*

$$d\mathbf{Y}_t = \mathcal{B}\mathbf{Y}_t dt + \mathbf{1}_q(\alpha_0 + \mathbf{a}^\top \mathbf{Y}_t) dS_t^{a,d}, \qquad t > 0,$$

*with inital value $\mathbf{Y}_0$, independent of $L$. Furthermore, $S^{a,d}$ is the fractional subordinator as defined in* (3.5)*. If the process $(\sigma_t^2)_{t\geq 0}$ is strictly stationary and non-negative almost surely, we define the FICOGARCH$(p,d,q)$ process $(G_t)_{t\geq 0}$ with parameters $\mathcal{B}, \mathbf{a}, \alpha_0$ and some initial value $G_0$ as the solution of the SDE*

$$dG_t = \sigma_{t-} dL_t, \qquad t > 0.$$

*The volatility process of the COGARCH(1,1) and also of the FICOGARCH$(1,d,1)$ is non-negative by definiton. This is not the case for the COGARCH$(p,q)$ model. Therefore, conditions as formulated in [3, Theorem 5.1] have to be considered to assure non-negativity of the volatility process.*

# References


[1] C. Bender, A. Lindner, and M. Schicks. Finite variation of fractional Lévy processes. *Journal of Theoretical Probability*, 25(2):594–612, 2012.

[2] C. Bender and T. Marquardt. Integrating volatility clustering into exponential Lévy models. *Journal of Applied Probability*, 46:609–628, 2009.

[3] P. Brockwell, E. Chadraa, and A. Lindner. Continuous-time GARCH processes. *The Annals of Applied Probability*, 16(2):790–826, 2006.

[4] P. Brockwell and T. Marquardt. Lévy-driven and fractionally integrated ARMA processes with continuous time parameter. *Statistica Sinica*, 15:477–494, 2005.

[5] L. Decreusefond and A. S. Üstünel. Stochastic analysis of the fractional Brownian motion. *Potential Analysis*, pages 177–214, 1999.

[6] S. Engelke and J. H. C. Woerner. A unifying approach to fractional Lévy processes. *Stochastics and Dynamics*, 13, 2013.

[7] C. Jost. Integral Transformations of Volterra Gaussian Processes. *Dissertation, University of Helsinki*, 2007.

[8] C. Klüppelberg, A. Lindner, and R. Maller. A continuous-time GARCH process driven by a Lévy process: stationarity and second-order behaviour. *Journal of Applied Probability*, 41(3):601–638, 2004.

[9] C. Klüppelberg, R. Maller, and A. Szimayer. The COGARCH: A review, with news on option pricing and statistical inference. In J. Blath, P. Imkeller, and S. Roelly, editors, *Surveys in Stochastic Processes. Proc. of the 33rd SPA Conference in Berlin*, pages 29–58. EMS Series of Congress Reports, EMS Publishing House, 2011.

[10] C. Klüppelberg and M. Matsui. Generalized fractional Lévy processes with fractional Brownian motion limit. *Adv. Appl. Prob.*, 47, 2015. To appear.

[11] B.B. Mandelbrot and J.W. van Ness. Fractional Brownian motions, fractional noises and applications. *SIAM Review*, 10(4):pp. 422–437, 1968.

[12] T. Marquardt. *Fractional Lévy Processes, CARMA Processes and Related Topics*. Dissertation, Technische Universität München, München, 2006.

[13] T. Marquardt. Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli*, 12(6):943–1126, 2006.

[14] M.M. Meerschaert and F. Sabzikar. Stochastic integration for tempered fractional Brownian motion. *Stoch. Proc. Appl.*, 124:2363–2387, 2014.

[15] Y. Mishura. *Stochastic Calculus for Fractional Brownian Motion and Related Processes*. Number 1929 in Lecture Notes in Mathematics. Springer, 2008.

[16] P.E. Protter. *Stochastic Integration and Differential Equations*. Springer, 2004.

[17] B.S. Rajput and J. Rosinski. Spectral representation of infinitely divisible processes. *Probability Theory and Related Fields*, 82:451–487, 1989.

[18] J. Rosinski. On path properties of certain infinitely divisible processes. *Stochastic Processes and their Applications*, 33:73–87, 1989.

[19] G. Samorodnitsky and M.S. Taqqu. *Stable Non-Gaussian Random Processes*. Chapman & Hall, 1994.

[20] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.

[21] H.J. Tikanmäki and Y. Mishura. Fractional Lévy processes as a result of compact interval integral transformation. *Stochastic Analysis and Applications*, 29:1081–1101, 2011.